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Diffusive Behavior of Asymmetric Zero-Range Processes

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We investigate a new interpretation for the Navier–Stokes corrections to the hydrodynamic equation of asymmetric interacting particle systems. We consider a system that starts from a measure associated with a profile that is constant along the drift direction. We show that under diffusive scaling the macroscopic behavior of the process is described by a nonlinear parabolic equation whose diffusion coefficient coincides with the diffusion coefficient of the hydrodynamic equation of the symmetric version of the process.

KEY WORDS: Asymmetric particle systems; Navier–Stokes corrections; relative entropy.

1. INTRODUCTION

A fundamental question in mathematical physics is the derivation and the interpretation of the Navier–Stokes equations. One of difficulties in the interpretation of this equation is that it is not scaling invariant and thus cannot be obtained by a scaling limit. Although this problem is still out of reach for Hamiltonian systems, important progress has been made recently in the context of interacting particle systems.[1, 3, 4, 7–10]

To fix ideas, consider the zero-range process evolving on the lattice $\mathbb{Z}^d$. This dynamics can be informally described as follows: fix a jump rate $g: \mathbb{N} \to \mathbb{R}_+$ such that $0 = g(0) < g(n)$ for $n \geqslant 1$ and a translation-invariant transition probability $p(x, y) = p(0, y-x) = p(y-x)$. If n particles are sitting on a site x, independently of what happens on the other sites, at rate $g(n)\, p(y)$ one particle jumps to site $x+y$. The configurations of the state space $\mathbb{N}^{\mathbb{Z}^d}$ are denoted by the Greek letter η, so that, for x in $\mathbb{Z}^d$,

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$\eta(x)$ represents the number of particles at site x for the configuration η. The total number of particles is the unique conserved quantity and for each density $\rho \geqslant 0$, there exists a translation-invariant product probability measure, denoted by ν_ρ, that is invariant for the dynamics and for which the density of particles is ρ.

The macroscopic evolution of the process under Euler rescaling is described[11] by the first-order quasilinear hyperbolic equation

$$\partial_t \rho + \gamma \cdot \nabla F(\rho) = 0 \tag{1.1}$$

where the smooth function F and $\gamma \in \mathbb{R}^d$ are parameters depending on the microscopic dynamics ($\gamma = \sum_z z p(z)$ and $F(\theta) = E_{\nu_\theta}[g(\eta(0))]$): assume that the system starts from a product measure with slowly varying density $\rho_0(\varepsilon u)$. Under Euler scaling (times of order $t\varepsilon^{-1}$) the density still has a slowly varying profile $\lambda_\varepsilon(t, \varepsilon u)$ that converges weakly (in fact pointwisely at every continuity point[6]) to the entropy solution of Eq. (1.1) with initial data ρ_0.

In the context of asymmetric interacting particle systems the Navier–Stokes equation takes the form

$$\partial_t \rho^\varepsilon + \gamma \cdot \nabla F(\rho^\varepsilon) = \varepsilon \sum_{i,j} \partial_{u_i}(D_{i,j}(\rho^\varepsilon)\, \partial_{u_j} \rho^\varepsilon) \tag{1.2}$$

where D is a diffusion coefficient. Three different interpretations have been proposed for the Navier–Stokes corrections:

(a) *The incompressible limit*[2,3]: Consider a small perturbation of a constant profile θ: $\rho_0^\varepsilon = \theta + \varepsilon a$. Assuming that this form persists at later times $[\rho^\varepsilon(t, u) = \theta + \varepsilon a(t, u)]$, we obtain from (1.2) the following equation for $a_\varepsilon = a(t\varepsilon^{-1}, u)$:

$$\partial_t a_\varepsilon + \varepsilon^{-1} F'(\theta)\, \gamma \cdot a_\varepsilon + (1/2)\, F''(\theta)\, \gamma \cdot \nabla a_\varepsilon^2 = D_{i,j}(\theta) \sum_{i,j} \partial^2_{u_i, u_j} a_\varepsilon + O(\varepsilon)$$

A Galilean transformation $m_\varepsilon(t, u) = a_\varepsilon(t, u + \varepsilon^{-1} t F'(\theta)\, \gamma)$ permits us to remove the diverging term of the last differential equation and to get a limit equation for $m = \lim_{\varepsilon \to 0} m_\varepsilon$,

$$\partial_t m + (1/2)\, F''(\theta)\, \gamma \cdot \nabla m^2 = D_{i,j}(\theta) \sum_{i,j} \partial^2_{u_i, u_j} m$$

(b) *First-order correction to the hydrodynamic equation*[1,7]: Fix a smooth profile ρ_0: $\mathbb{R}^d \to \mathbb{R}_+$ and consider a process starting from a product measure with slowly varying density $\rho_0(\varepsilon u)$. We have seen that under Euler scaling the density is still a slowly varying profile $\lambda_\varepsilon(t, \varepsilon u)$ that converges weakly to the entropy solution of Eq. (1.1) with initial data ρ_0. This second interpretation asserts that the solution of Eq. (1.2) with initial profile ρ_0 approximates λ_ε up to the order ε:

$$\varepsilon^{-1}(\lambda_\varepsilon - \rho^\varepsilon) \to 0$$

in a weak sense as $\varepsilon \downarrow 0$.

(c) *Long-time behavior*[7]: The third interpretation consists in analyzing the behavior of the solution of Eq. (1.2) on time scales of order $t\varepsilon^{-1}$. Let $b_\varepsilon(t, u) = \rho(t\varepsilon^{-1}, u)$. From (1.2) we obtain the following equation for b_ε:

$$\partial_t b_\varepsilon + \varepsilon^{-1}\gamma \cdot \nabla F(b_\varepsilon) = \sum_{i,j} \partial_{u_i}(D_{i,j}(b_\varepsilon)\, \partial_{u_j} b_\varepsilon)$$

To eliminate the diverging term $\varepsilon^{-1}\gamma \cdot \nabla F(b_\varepsilon)$, assume that the initial data (and therefore the solution at any fixed time) are constant along the drift direction: $\gamma \cdot \nabla \rho_0 = 0$. In this case we get the parabolic equation

$$\partial_t b_\varepsilon = \sum_{i,j} \partial_{u_i}(D_{i,j}(b_\varepsilon)\, \partial_{u_j} b_\varepsilon)$$

which describes the evolution of the system in the hyperplane orthogonal to the drift.

Notice that while the first and third interpretations concern the behavior of the system under diffusive rescaling, the second one is a statement on the process under Euler rescaling. Interpretations (a) and (b) have been proved for asymmetric simple exclusion processes in dimensions $d \geqslant 3$[3, 7] and a double variational formula for the diffusion coefficient was deduced. As one would expect, the diffusion coefficients of the two interpretations are the same and[9] may be expressed by a Green–Kubo formula. It was also proved (Corollary 6.2, ref. 8) that the diffusion coefficient is strictly bounded below in the matrix sense by the diffusion coefficient that governs the evolution of the symmetric process.

In contrast with interpretations (a) and (b), the third one is meaningful in dimension $d \geqslant 2$. However, since the initial profile is constant along the drift direction, this third interpretation gives no information on the drift in this direction. Moreover, as noticed by S. Olla, the zero-range dynamics is such that particles do not feel the gradient of the profile. It is therefore not surprising that for zero-range processes the diffusion coefficient, at least in the direction orthogonal to the drift, is diagonal and equal to the diffusion coefficient of the symmetric process.

The purpose of this paper is to give a rigorous proof of the third interpretation in the case of asymmetric zero-range processes and prove that the diffusion coefficient of the Navier–Stokes corrections coincides in this case with the diffusion coefficient of the hydrodynamic equation of the symmetric process under diffusive scaling. The approach adopted to prove this result is based on the relative entropy method introduced by Yau.[12]

The third interpretation relies on the following observation. We have already mentioned above that, in the interacting particle context, the Euler equation reduces to a nonlinear first-order hyperbolic equation

$$\partial_t \rho + \gamma \cdot \nabla F(\rho) = 0$$

In finite volume with periodic boundary conditions, say on the torus $[0, 1]^d$, asymptotically as $t \uparrow \infty$, the entropic solution $\rho(t, u)$ converges to a stationary solution which is constant along the drift:

$$\rho(t, \cdot) \to \rho_{\infty}(u) = \int_0^1 \rho_0(u + r\gamma)\, dr$$

provided ρ_0 stands for the initial data. In particular, if we consider the asymptotic process under diffusive rescaling, we expect it to become immediately constant along the drift direction:

$$\gamma \cdot \nabla \lim_{N \to \infty} \mathbb{E}_{\mu^N}[\eta_{tN^2}([uN])] = 0$$

for every $t > 0$ and for any initial profile. Here $\mathbb{E}_{\mu^N}$ stands for the expectation with respect to the probability measure on the path space $D(\mathbb{R}_+, \mathbb{N}^{\mathbb{Z}^d})$ induced by the zero-range dynamics described above and a product measure μ^N with slowly varying parameter. In contrast, on the hyperplane orthogonal to the drift, the profile should evolve smoothly in time according to a parabolic equation.

2. NOTATION AND RESULTS

We consider particles on the discrete d-dimensional torus $\mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$ moving according to a transition probability p on $\mathbb{T}_N^d$ satisfying the following two conditions:

- p is shift invariant: $p(x, y) = p(0, y - x) := p(y - x)$.
- p has a finite range: $p(x) = 0$ if $|x| > K$.

We denote by m the mean of the law p and by σ the matrix of its second moments and we suppose that σ is positive definite.

The system evolves with a zero-range dynamics, that is, the jump rate of a particle located on some site only depends on the number of particles on this site. Let

$$\eta_t \in \mathbb{N}^{\mathbb{T}_N^d} := X_N^d$$

be the configuration of particles at time t, namely for all $x \in \mathbb{T}_N^d$, $\eta_t(x)$ is the number of particles on site x; then (η_t) is the Markov process on X_N^d with generator L_N defined for functions f on X_N^d by

$$L_N f(\eta) = \sum_{x,y} p(y-x)\, g(\eta(x))[f(\eta^{x,y}) - f(\eta)]$$

where $\eta^{x,y}$ is the configuration obtained after the jump of a particle from site x to site y. The jump rate function g from $\mathbb{N}$ to $\mathbb{R}_+$ is such that $g(0)=0$ and $\sup_n(g(n+1)-g(n))<\infty$. We denote by (S_t^N) the semigroup associated to the generator L_N.

The product probability $\bar{\nu}_\varphi^N$ on X_N^d, whose marginal laws are defined by

$$\bar{\nu}_\varphi^N(\eta(x)=k) = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g(k)!}$$

is invariant for the zero-range process. In the previous formula, $g(k)!$ stands for the product $g(1)\cdots g(k)$ and $Z(\varphi)$ is a normalization constant. If we denote by φ^* the radius of convergence of the entire function $\sum_k \varphi^k/g(k)!$, we assume that

$$\lim_{\varphi \uparrow \varphi^*} Z(\varphi) = \infty$$

It can be easily shown that, under this condition, $\eta(x)$ has some exponential moments: for every $\varphi < \varphi^*$, there exists a positive constant $\theta = \theta(\varphi)$ such that

$$\bar{\nu}_\varphi^N[\exp \theta\eta(x)] < \infty \tag{2.1}$$

In order to prove a one-block estimate, we impose some conditions on the jump rate $g(\cdot)$. We assume that either $\eta(0)$ has all exponential moments finite under any measure $\bar{\nu}_\varphi^N$ [this is equivalent to requiring the partition function $Z(\cdot)$ to be finite on $\mathbb{R}_+$ because $\bar{\nu}_\varphi^N[\exp \theta\eta(x)] = Z(\varphi e^\theta)/Z(\varphi)$]:

(FEM) $Z(\varphi) < \infty$ for all φ in $\mathbb{R}_+$

or that g increases slower than linearly:

(SLG) $\limsup_{k\to\infty}(g(k)/k)=0$

Notice that in the case where the rate jump is nondecreasing, at least one of these assumptions is satisfied.

Let Φ be the increasing function such that $\bar{\nu}^N_{\Phi(\rho)}[\eta(x)]=\rho$. Notice that Φ is an infinitely differentiable Lipschitz function on $\mathbb{R}_+$. We denote by ν^N_ρ the measure $\bar{\nu}^N_{\Phi(\rho)}$. In the case where ρ is a function on $\mathbb{T}^d_N$, ν^N_ρ is the measure on $\mathbb{T}^d_N$ with the macroscopic density ρ, that is, the product measure whose marginals are given by

$$\nu^N_\rho(\eta(x)=k)=\nu^N_{\rho(x/N)}(\eta(0)=k)$$

For a continuous function ρ_0 on $\mathbb{T}^d$, consider such a sequence $(\nu^N_{\rho_0})$ of product initial distributions associated with the profile ρ_0. In the case of a nondecreasing jump rate $g(\cdot)$, Rezakhanlou proved that the hydrodynamic behavior of the system under Euler scaling is described by the unique entropic solution of the equation

$$\begin{cases}\partial_t\rho=m\cdot\nabla\Phi(\rho)\\ \rho(0,\cdot)=\rho_0(\cdot)\end{cases}$$

It is well known that as time increases, the solution $\rho(t,\cdot)$ converges to a stationary profile $\rho_\infty(\cdot)$ which is constant along the drift direction. More precisely, denote by $\tilde{m}$ the normalized drift $m/\|m\|_2$ (where $\|m\|_2$ is the Euclidean norm of m) and define $\rho_\infty:\mathbb{T}^d\to\mathbb{R}_+$ as

$$\rho_\infty(u)=\int_0^1\rho_0(u+\lambda\tilde{m})\,d\lambda$$

Then, as $t\uparrow\infty$, $\rho(t,\cdot)$ converges to $\rho_\infty(\cdot)$ in $L^1(\mathbb{T}^d)$.

In view of this asymptotic behavior, we investigate here the asymmetric zero-range process under diffusive rescaling starting from an initial density profile constant along the drift direction m: for any $\lambda\in\mathbb{R}$,

$$\rho_0(u+\lambda m)=\rho_0(u) \tag{2.2}$$

We prove that the macroscopic behavior of the zero-range process under diffusive rescaling (η_{tN^2}) is described by the solution of the parabolic nonlinear equation

$$\begin{cases}\partial_t\rho=\Delta_\sigma\Phi(\rho)\\ \rho(0,\cdot)=\rho_0(\cdot)\end{cases} \tag{2.3}$$

where Δ_σ is the second-order differential operator related to the covariance matrix σ: if we denote by d^2G the matrix of the second derivatives of a function G on $\mathbb{R}^d$ and if X is a random vector with distribution p, then

$$\Delta_\sigma G = (\nabla^* \sigma \nabla)\, G = E(X^* d^2 G X) \tag{2.4}$$

Before describing the strategy of the proof, we present some heuristic arguments that led to (2.3). For the zero-range process, the microcurrent reads

$$d\eta_t(x) = N^2 L_N \eta_t(x) = N^2 \sum_y p(-y)\{ g(\eta_t(x+y)) - g(\eta_t(x))\}$$

By the local equilibrium assumption, on the average this expression is equal to

$$\begin{aligned} N^2 \sum_y p(-y)\{ &\Phi(\rho(t, (x+y)/N)) - \Phi(\rho(t, x/N))\} \\ &= N \sum_y p(-y)\; y \cdot \nabla \Phi(\rho(t, x/N)) \\ &\quad + (1/2) \sum_y p(y)\; y^* d^2 \Phi(\rho(t, x/N))\; y + o_N(1) \end{aligned}$$

Since for every $t \geqslant 0$, $\rho(t, \cdot)$ is constant along the drift direction, the first term vanishes and we obtain (2.3).

The method we use was introduced by Yau[12] and may be described as follows. Consider a sequence of initial measures μ^N whose entropy relative to $\nu^N_{\rho_0}$, normalized by the volume, vanishes as $N \uparrow \infty$. It consists in proving that, at any time t, the normalized relative entropy of the process law at time tN^2 with respect to the measure $\tilde{\nu}^N_t := \nu^N_{\rho(t, \cdot)}$, where $\rho(t, x)$ is the solution of (2.3), goes to zero as N goes to infinity. More precisely, let $\rho_0 \in C^{2+\kappa}(\mathbb{T}^d)$ $(0 < \kappa < 1)$ be a strictly positive smooth function satisfying (2.2). We denote by $\rho(t, x)$ the smooth solution of (2.3) in $C^{1, 2+\kappa}(\mathbb{R}_+ \times \mathbb{T}^d)$. By the maximum principle, we have

$$\inf_{t \in [0, T]} \inf_{u \in \mathbb{T}^d} \rho(t, u) = \inf_{u \in \mathbb{T}^d} \rho_0(u) > 0$$

$$\sup_{t \in [0, T]} \sup_{u \in \mathbb{T}^d} \rho(t, u) = \sup_{u \in \mathbb{T}^d} \rho_0(u) < \infty$$

We suppose that the sequence μ^N of initial distributions for the zero-range process satisfies the entropy condition

$$H(\mu^N \mid \tilde{\nu}^N_0) = o(N^d) \tag{2.5}$$

and we denote by μ_t^N the law $\mu^N S_{tN^2}^N$ of the speeded-up zero-range process at time t. Notice that by the explicit formula for the entropy and the entropy inequality, for every $\alpha > 0$,

$$H(\mu^N \mid \nu_\alpha^N) \leqslant (1+\gamma^{-1})\, H(\mu^N \mid \tilde{\nu}_0^N) + \gamma^{-1} \log \int \left(\frac{d\tilde{\nu}_0^N}{d\nu_\alpha^N}\right)^\gamma d\tilde{\nu}_0^N$$

for every $\gamma > 0$. Therefore, choosing γ small enough and keeping in mind the estimate (2.1), we get that

$$H(\mu^N \mid \nu_\alpha^N) \leqslant K_1 N^d \tag{2.6}$$

for some finite constant K_1.

Theorem 2.1. Under hypotheses (2.2), (2.5), and (SLG) or (FEM), for any $t \geqslant 0$,

$$H(\mu_t^N \mid \tilde{\nu}_t^N) = o(N^d)$$

Then we obtain the following result.

Corollary 2.2. Under the assumptions of Theorem 2.1, for any bounded cylinder function Ψ: $X_N^d \to \mathbb{R}$ (Ψ just depends on η through a finite number of sites) and for any continuous function F on $\mathbb{T}^d$, we have

$$\lim_{N\to\infty} \int \left| N^{-d} \sum_x F(x/N)\, \tau_x \Psi(\eta) - \int F(u)\, E_{\nu_{\rho(t,u)}}[\Psi]\, du \right| d\mu_t^N = 0$$

where $\rho(t, u)$ is the solution of (2.3).

In the case where the zero-range process is attractive, the conservation of local equilibrium can be deduced from Corollary 2.2 using results of ref. 6.

Corollary 2.3. Assume that the jump rate function g is nondecreasing and that the initial distribution μ^N is a product measure associated to a profile ρ_0 of class $C^{2+\kappa}(\mathbb{T}^d)$. Under the assumptions of Theorem 2.1, for every bounded cylinder function Ψ

$$\lim_{N\to\infty} \mu_t^N \tau_{[uN]}[\Psi] = \nu_{\rho(t,u)}[\Psi]$$

where $\rho(t, u)$ is the solution of (2.3).

3. PROOFS

Proof of Theorem 2.1. Fix some $\alpha > 0$ and denote by ψ_t^N the density of the measure $\tilde{\nu}_t^N$ with respect to ν_α^N. We have

$$\psi_t^N(\eta) = \exp\left\{\sum_x [\eta(x) \log \Phi_\alpha(\rho(t, x/N)) - \log Z_\alpha(\Phi(\rho(t, x/N)))]\right\} \tag{3.1}$$

where Φ_α and Z_α are given by

$$\Phi_\alpha(\mu) = \frac{\Phi(u)}{\Phi(\alpha)}, \qquad Z_\alpha(u) = \frac{Z(\Phi(u))}{Z(\Phi(\alpha))}$$

We define f_t^N as the density of μ_t^N with respect to ν_α^N. Let L_N^* be the adjoint operator of L_N in $L^2(\nu_\alpha^N)$, that is, the Markov generator defined by

$$L_N^* f(\eta) = \sum_{x, y} p(x-y)\, g(\eta(x))[f(\eta^{x, y}) - f(\eta)]$$

and let (S_t^{N*}) be the semigroup related to L_N^*; then $f_t^N = S_{tN^2}^{N*} f_0^N$ and $\partial_t f_t^N = N^2 L_N^* f_t^N$.

The relative entropy $H_N(t) := H(\mu_t^N \mid \tilde{\nu}_t^N)$ is given by

$$\begin{aligned} H_N(t) &= \int \frac{d\mu_t^N}{d\tilde{\nu}_t^N} \log \frac{d\mu_t^N}{d\tilde{\nu}_t^N}\, d\tilde{\nu}_t^N \\ &= \int f_t^N \log \frac{f_t^N}{\psi_t^N}\, d\nu_\alpha^N \end{aligned}$$

Using the Gronwall lemma, we will establish Theorem 2.1 if we prove that there is a positive constant γ such that

$$H_N(t) \leqslant o(N^d) + \gamma^{-1} \int_0^t H_N(s)\, ds \tag{3.2}$$

The proof of this inequality is divided into several lemmas. We start with an estimate of the relative entropy production.

Lemma 3.1. For any $t \geqslant 0$

$$\partial_t H_N(t) \leqslant \int \frac{1}{\psi_t^N} \{N^2 L_N^* \psi_t^N - \partial_t \psi_t^N\}\, f_t^N\, d\nu_\alpha^N$$

Proof. As f_t^N is a solution of $\partial_t f_t^N = N^2 L_N^* f_t^N$ and as $\rho(t, \cdot)$ is a smooth function in t,

$$\partial_t H_N(t) = \int N^2 L_N^* f_t^N \cdot \log\left[\frac{f_t^N}{\psi_t^N}\right] d\nu_\alpha^N$$
$$+ \int \left\{N^2 L_N^* f_t^N - f_t^N \frac{\partial_t \psi_t^N}{\psi_t^N}\right\} d\nu_\alpha^N \tag{3.3}$$

Since L_N^* is the adjoint operator of L_N in $L^2(\nu_\alpha^N)$, we have

$$\int L_N^* f_t^N \, d\nu_\alpha = 0$$

and the first term of the right-hand side of (3.3) can be written as

$$\int \psi_t^N \frac{f_t^N}{\psi_t^N} N^2 L_N \left(\log \frac{f_t^N}{\psi_t^N}\right) d\nu_\alpha^N$$

Moreover, for all positive numbers a and b, the basic inequality

$$a[\log b - \log a] \leqslant (b - a)$$

shows that for any positive function h

$$h L_N(\log h) \leqslant L_N h$$

So the last integral is bounded above by

$$\int \psi_t^N N^2 L_N \left(\frac{f_t^N}{\psi_t^N}\right) d\nu_\alpha = \int \frac{f_t^N}{\psi_t^N} N^2 L_N^* \psi_t^N \, d\nu_\alpha \quad \blacksquare$$

On the one hand, we observe from formula (3.1) that

$$\frac{\partial_t \psi_t^N}{\psi_t^N} = \sum_x \left\{\eta(x) \frac{\partial_t \Phi(\rho(t, x/N))}{\Phi(\rho(t, x/N))} - \frac{\partial_t \Phi(\rho(t, x/N))\, Z'(\Phi(\rho(t, x/N)))}{Z(\Phi(\rho(t, x/N)))}\right\}$$

A simple computation on the partition function Z shows that for any non-negative number λ,

$$\frac{Z'(\Phi(\lambda))}{Z(\Phi(\lambda))} = \frac{\lambda}{\Phi(\lambda)}$$

Therefore, since ρ is the solution of (2.3),

$$\frac{\partial_t \psi_t^N}{\psi_t^N} = \sum_x G(t, x/N)(\eta(x) - \rho(t, x/N))\, \Phi'(\rho(t, x/N)) \tag{3.4}$$

where

$$G(t, x/N) := \frac{\Delta_\sigma \Phi(\rho)(t, x/N)}{2\Phi(\rho(t, x/N))} \tag{3.5}$$

On the other hand, a straightforward computation shows that

$$\frac{N^2 L_N^* \psi_t^N}{\psi_t^N} = N^2 \sum_{x, y} p(x-y)\, g(\eta(x)) \left\{ \frac{\Phi(\rho(t, y/N))}{\Phi(\rho(t, x/N))} - 1 \right\} \tag{3.6}$$

Now, using the Taylor expansion, we obtain

$$\begin{aligned} &\sum_y p(x-y)\{\Phi(\rho(t, y/N)) - \Phi(\rho(t, x/N))\} \\ &\quad = \frac{-1}{N} \left(\sum_z z p(z) \right) \cdot \nabla \Phi(\rho(t, x/N)) \\ &\qquad + \frac{1}{2N^2} \sum_z p(z)\, z^* d^2 \Phi(\rho(t, x/N))\, z + o(N^{-2}) \end{aligned}$$

It results from assumption (2.2) that

$$\left(\sum_z z p(z) \right) \cdot \nabla \Phi(\rho(t, x/N)) = m \cdot \nabla \rho(t, x/N)\, \Phi'(\rho(t, x/N)) = 0$$

and from definition (2.4) that

$$\sum_z p(z)\, z^* d^2 \Phi(\rho(t, x/N))\, z = \Delta_\sigma \Phi(\rho(t, x/N))$$

Therefore,

$$\frac{N^2 L_N^* \psi_t^N}{\psi_t^N} = \sum_x g(\eta(x))\, G(t, x/N) + o_N(1) \sum_x g(\eta(x)) \tag{3.7}$$

where G is the function defined in (3.5). Notice that

$$\sum_x \Delta_\sigma \Phi(\rho(t, x/N)) = o(N^d) \tag{3.8}$$

because the left-hand side divided by N^d vanishes as $N \uparrow \infty$. Thus, from Lemma 3.1, (3.4), (3.7), and (3.8), we get that the time derivative of $H_N(t)$ is bounded above by

$$\int \sum_x G(t, x/N)\{ g(\eta(x)) - \Phi(\rho(t, x/N)) - \Phi'(\rho(t, x/N))(\eta(x) - \rho(t, x/N))\} f_t^N \, d\nu_\alpha^N + o_N(1) \int \sum_x g(\eta(x)) \, f_t^N \, d\nu_\alpha^N + o(N^d) \tag{3.9}$$

Using the entropic inequality, we have

$$\int \sum_x g(\eta(x)) \, f_t^N \, d\nu_\alpha^N \leqslant \frac{1}{\theta} H(\mu_t^N \mid \tilde{\nu}_t^N) + \frac{1}{\theta} \log \int \exp\left[\theta \sum_x g(\eta(x)) \right] d\tilde{\nu}_t^N$$

for every $\theta > 0$. Since by assumption $\sup_n (g(n+1) - g(n)) < \infty$, there exists a finite constant K_0 such that $g(k) \leqslant K_0 k$ for all k in $\mathbb{N}$. Therefore, the right-hand side of this expression is bounded above by

$$\frac{1}{\theta} H_N(t) + \frac{N^d}{\theta} \log \int \exp[\theta K_0 \eta(0)] \, d\nu_{\rho^*}^N$$

where $\rho^* = \sup_u \rho_0(u)$ because the family of invariant measures $\{\nu_\alpha^N, \alpha \geqslant 0\}$ is increasing in α. For θ small enough, by (2.1), the second term is bounded by $N^d C(\rho^*, K_0)$. Hence,

$$o_N(1) \int \sum_x g(\eta(x)) \, f_t^N \, d\nu_\alpha^N \leqslant C(\rho^*, K_0) \, H_N(t) + o(N^d) \tag{3.10}$$

We now integrate in time inequality (3.9) and in view of (2.6) we can use the well-known one-block estimate (see, e.g., ref. 5) to replace the local function $g(\eta(x))$ by its average on a large microscopic block with respect to the invariant measure with parameter equal to the density of particles in this block:

Lemma 3.2. Under the assumptions of Theorem 2.1, for any $t > 0$ and for any continuous function F on $[0, t] \times \mathbb{T}^d$,

$$\lim_{l \to \infty} \lim_{N \to \infty} \int_0^t ds \int \frac{1}{N^d} \sum_x F(s, x/N)\{ g(\eta(x)) - \Phi(\eta_s^l(x))\} \, f_s^N \, d\nu_\alpha^N = 0$$

where

$$\eta_s^l(x) = \frac{1}{(2l+1)^d} \sum_{|y-x| \leq l} \eta(y)$$

Replacing now $\eta(x)$ by a local average in a large microscopic block, we obtain from inequality (3.9), estimate (3.10), and Lemma 3.2 that the entropy $H_N(t)$ is bounded above by

$$\begin{aligned}
&\int_0^t ds \int \sum_x G(s, x/N)\{\Phi(\eta_s^l(x)) - \Phi(\rho(s, x/N)) \\
&\quad - \Phi'(\rho(s, x/N))(\eta_s^l(x) - \rho(s, x/N))\} \, f_s^N \, d\nu_\alpha^N \\
&\quad + H_N(0) + C(\rho^*, K_0) \int_0^t ds \, H(\mu_s^N \mid \tilde{\nu}_s^N) + o(N^d) + R(l, N^d)
\end{aligned}$$

with

$$\lim_{l \to \infty} \lim_{N \to \infty} N^{-d} R(l, N^d) = 0$$

From assumption (2.5), $H_N(0)$ is bounded by $o(N^d)$. So applying now the entropic inequality, we obtain that this last expression is less than or equal to

$$\begin{aligned}
&\frac{1}{\gamma} \int_0^t ds \log \int \exp\Big[\gamma \sum_x G(s, x/N)\{\Phi(\eta_s^l(x)) - \Phi(\rho(s, x/N)) \\
&\quad - \Phi'(\rho(s, x/N))(\eta_s^l(x) - \rho(s, x/N))\} \Big] \, d\tilde{\nu}_s^N \\
&\quad + o(N^d) + R(l, N^d) + (C(\rho^*, K_0) + \gamma^{-1}) \int_0^t ds \, H_N(s)
\end{aligned}$$

for any positive constant γ. To conclude the proof of (3.2), it remains to show that there exists a positive constant γ_0 such that, for any $0 \leq s \leq t$,

$$\begin{aligned}
&\limsup_{l \to \infty} \limsup_{N \to \infty} \frac{1}{N^d} \log \int \exp\Big[\gamma_0 \sum_x H(s, x/N)\{\Phi(\eta^l(x)) - \Phi(\rho(s, x/N)) \\
&\quad - \Phi'(\rho(s, x/N))[\eta^l(x) - \rho(s, x/N)]\} \Big] \, d\tilde{\nu}_s^N \leq 0
\end{aligned}$$

This inequality is proved in ref. 5 using a large-deviation principle for $\eta^l(x)$ under $\tilde{\nu}_s^N$. ∎

Proof of Corollary 2.2. Let Ψ be a bounded cylinder function on X_N^d. Since $F(\cdot)$ and $\rho(t, \cdot)$ are continuous and since Ψ is bounded, the result will be proved if we show that

$$\limsup_{l \to \infty} \limsup_{N \to \infty} \int N^{-d} \sum_x \left| (2l+1)^{-d} \sum_{|y-x| \leqslant l} \tau_y \Psi(\eta) - E_{\nu_{\rho(t, x/N)}}[\Psi] \right| d\mu_t^N \leqslant 0$$

To keep notation simple, we shall assume that Ψ depends on the configuration only through site 0: $\Psi(\eta) = \Psi(\eta(0))$. Because of the entropic inequality, for any $\gamma > 0$, the last integral is bounded above by

$$\frac{1}{\gamma N^d} H(\mu^N S_t^N \mid \tilde{\nu}_t^N) + \frac{1}{\gamma N^d} \log \int \exp \left[\gamma \sum_x \left| (2l+1)^{-d} \right.\right.$$
$$\left.\left. \times \sum_{|y-x| \leqslant l} \Psi(\eta(y)) - E_{\nu_{\rho(t, x/N)}}[\Psi] \right| \right] d\tilde{\nu}_t^N \tag{3.11}$$

We will choose γ as a function of l. By Theorem 2.1, the first term goes to 0 as $N \uparrow \infty$. Moreover, since $\tilde{\nu}_t^N$ is a product measure, the random variables $(2l+1)^{-d} \sum_{|y-x_1| \leqslant l} \Psi(\eta(y))$ and $(2l+1)^{-d} \sum_{|y-x_2| \leqslant l} \Psi(\eta(y))$ are independent as soon as $|x_1 - x_2| > 2l$. So, applying the Hölder inequality, we find that the right-hand side of (3.11) is less than

$$\frac{1}{\gamma N^d} \sum_x \frac{1}{(2l+1)^d} \log \int \exp \left[\gamma \left| \sum_{|y-x| \leqslant l} \Psi(\eta(y)) - E_{\nu_{\rho(t, x/N)}}[\Psi] \right| \right] d\tilde{\nu}_t^N$$

Since the density profile $\rho(t, \cdot)$ is continuous, as N goes to ∞, this sum goes to

$$\int_{\mathbb{T}^d} du \frac{1}{\gamma(2l+1)^d} \log \int \exp \left[\gamma \left| \sum_{|y| \leqslant l} \Psi(\eta(y)) - E_{\nu_{\rho(t,u)}}[\Psi] \right| \right] d\nu_{\rho(t,u)}$$

From inequalities $e^x \leqslant 1 + x + 2^{-1} x^2 e^x$ and $\log(1+x) \leqslant x$, this integral is bounded above by

$$\int_{\mathbb{T}^d} du \frac{1}{\gamma(2l+1)^d} \left\{ \gamma \int \left| \sum_{|y| \leqslant l} \Psi(\eta(y)) - E_{\nu_{\rho(t,u)}}[\Psi] \right| d\nu_{\rho(t,u)} \right.$$
$$\left. + 2\gamma^2 (2l+1)^{2d} \|\Psi\|_\infty^2 \exp[2\gamma(2l+1)^d \|\Psi\|_\infty] \right\}$$

To conclude, it is enough to set $\gamma = (2l+1)^{-d} \varepsilon$, apply the law of large numbers for $l \uparrow \infty$, and let ε go to 0. ∎

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